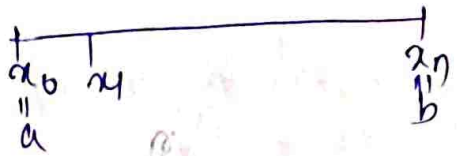
## Newton-cotes Rule :-

Let $I(f) = \int_a^b f(x)\,dx$




Divide $[a,b]$ into $n$ equal parts.

i.e $x_0 = a, x_1, x_2, \ldots, x_{n-1}, x_n = b$, where $x_i - x_{i-1} = h$.

Now To calculate $I(f)$ we use the Lagrange Interpolating polynomial

$$P_n(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + \cdots + l_n(x) f(x_n)$$

$$y(x) \approx P_n(x), \quad a \le x \le b,$$

So $$I(f) = \int_a^b f(x)\,dx \simeq \int_a^b P_n(x)\,dx = I(P)$$

$$= f(x_0)\int_a^b l_0(x)\,dx + \cdots + f(x_n)\int_a^b l_n(x)\,dx$$

Let $w_i = \int_a^b l_i(x)\,dx$, $i = 0(1)n$.

So $$\boxed{I(P_n) = w_0 f_0 + w_1 f_1 + \cdots w_n f_n = R_{n+1}(f) \ (\text{say})} \quad \text{---(1)}$$

The rule $R_{n+1}(f)$ is called the (n+1) point Newton-cotes quadrature formula.

involving $n+1$ nodes $x_0, x_1, \ldots x_n$ and $n+1$ coefficients or weights $w_0, w_1, \ldots w_n$.

To calculate $w_i = \int_a^b l_i(x)\,dx$

$$= \int_a^b \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}\,dx$$

Let where $x_i = x_0 + ih$

$x = x_0 + \alpha h$

$0 \le \alpha \le n$

$$dx = h\,d\alpha$$

when $x \to a$, $\alpha \to 0$

$x \to b$, $\alpha \to n$

$$\Rightarrow w_i = \int_0^n \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-i+1)(\alpha-i-1)\cdots(\alpha-n)}{i(i-1)(i-2)\cdots(-1)\cdots(i-n)}\, h\,d\alpha$$

$$= \frac{h}{i!(-1)^{i-n}(n-i)!}\int_0^n \alpha(\alpha-1)\cdots(\alpha-i+1)(\alpha-i-1)\cdots(\alpha-n)\,d\alpha$$

$$= \frac{h(-1)^{n-i}}{i!\,(n-i)!}\int_0^n \alpha(\alpha-1)\cdots(\alpha-i+1)(\alpha-i-1)\cdots(\alpha-n)\,d\alpha \quad \text{---------- (2).}$$

where $i = 0(1)n$.

Error in Newton codes quadrature

We know $f(x) = P_n(x) + \frac{(x-x_0)\cdots(x-x_n)}{(n+1)!} f^{n+1}(\eta)$

$\eta \in (a,b)$

for $f \in C^{n+1}[a,b]$.

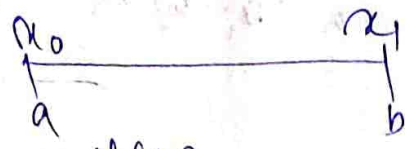
$E(f) = R_{n+1}(f) + E(f)$

So $E(f) = \frac{1}{(n+1)!}\int_a^b (x-x_0)\cdots(x-x_n) f^{n+1}(\eta)\,dx.$

$n=1$ (Trapizoidal rule)

Here no. of node is 2 i.e $x_0, x_1$

$x_0 = a$, $x_1 = b$.




So $I(f) = R_2(f) = w_0 f(a) + w_1 f(b)$

$$w_0 = \int_a^b l_0(x)\,dx$$

$$= \int_a^b \frac{x-x_1}{x_0-x_1}\,dx$$

putting $x = x_0 + \alpha h$, $x_1 = x_0 + h$

$$= \int_0^1 \frac{(\alpha-1)h}{-h}\,h\,d\alpha$$

$$= -h\int_0^1 (\alpha-1)\,d\alpha = -h\left[\frac{\alpha^2}{2} - \alpha\right]_0^1$$

$$= -h\left(\frac{1}{2}-1\right) = -h\left(-\frac{1}{2}\right)$$

$$\Rightarrow w_0 = \frac{h}{2}$$

$$w_1 = \int_a^b l_1(x)\,dx$$

$$= \int_a^b \frac{x-x_0}{x_1-x_0}\,dx$$

$$= \int_0^1 \frac{\alpha h}{h}\,h\,d\alpha = h\int_0^1 \alpha\,d\alpha$$

$$= h\left[\frac{\alpha^2}{2}\right]_0^1 = h\,\frac{1}{2}$$

$$\Rightarrow w_1 = \frac{h}{2}$$

$$I(f) = \frac{h}{2}f(a) + \frac{h}{2}f(b)$$

$$= \frac{h}{2}\left[f(a) + f(b)\right]$$

$$\Rightarrow \boxed{I(f) = \frac{b-a}{2}\left[f(a)+f(b)\right] = T(f)} \quad (\because h = b-a) \qquad \text{---(3)}$$

Error in trapezoidal rule.

$$E = \frac{1}{2!}f''(\eta)\int_a^b (x-x_0)(x-x_1)\,dx \quad ; \eta \in (x_0, x_1)$$

$$\Rightarrow E = \frac{1}{2} f''(\eta) \int_a^b (x-a)(x-b)\, dx$$

$$= \frac{f''(\eta)}{2} \int_a^b \{x^2 - (a+b)x + ab\}\, dx$$

$$= \frac{f''(\eta)}{2} \left[\frac{x^3}{3} - (a+b)\frac{x^2}{2} + abx\right]_a^b$$

$$= \frac{f''(\eta)}{2} \left(\frac{b^3}{3} - (a+b)\frac{b^2}{2} + ab^2 - \frac{a^3}{3} + (a+b)\frac{a^2}{2} - a^2 b\right)$$

$$= \frac{f''(\eta)}{2} \frac{2(b^3-a^3) + 3(a+b)(a^2-b^2) + 6ab(b-a)}{6}$$

$$= \frac{f''(\eta)}{2} \left[\frac{-(b^3 + 3a^2b - 3ab^2 - a^3)}{6}\right]$$

$$\boxed{E(f) = -\frac{1}{12}(b-a)^3 f''(\eta).} \quad ; \quad a \le \eta \le b$$

(4)